Soft marginal primal

$$\inf_{\gamma \in \mathcal{M}(X \times X)} F(A\gamma) + G(\gamma)$$

G: as before

$A = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ as before

$$F(\sigma_1, \sigma_2) = \varkappa \, KL(\sigma_1 | \mu) + \varkappa \, KL(\sigma_2 | \nu)$$

Conjugate of F:

$$F^*(\phi, \psi) = \varkappa \left[KL^*(\phi/\varkappa \,|\, \mu) + KL^*(\psi/\varkappa \,|\, \nu) \right]$$

dual problem

recall: $KL^*(\phi | \mu) = \int [\exp(\phi) - 1] \, d\mu$

$$\sup_{\phi, \psi} -F^*(-(\phi, \psi)) - \underbrace{G^*(A^*(\phi, \psi))}_{\text{as before}}$$

$$= \sup_{\phi, \psi} \varkappa \left[\int \left[1 - \exp(-\phi/\varkappa)\right] d\mu + \int \left[1 - \exp(-\psi/\varkappa)\right] d\nu \right]$$

$$- \varepsilon \int \left[\exp\left(\frac{\phi \oplus \psi - c}{\varepsilon}\right) - 1\right] d\mu \otimes \nu$$

recall dual:

$$\sup_{\phi,\psi} \varkappa\left[\int\left[1-\exp(-\phi/\varkappa)\right]d\mu + \int\left[1-\exp(-\psi/\varkappa)\right]d\nu\right]$$

$$- \varepsilon \int \left[\exp\left(\frac{\phi \oplus \psi - c}{\varepsilon}\right) - 1\right] d\mu \otimes \nu$$

alternating maximization / opt. condition:

$$\frac{d\, J(\phi + t\cdot\eta, \psi)}{dt}\Bigg|_{t=0} = \int \exp\left(-\frac{\phi}{\varkappa}\right)\eta\, d\mu - \int \exp\left(\frac{\phi}{\varepsilon}\right)\left[\int \exp\left(\frac{\psi - c(\dots)}{\varepsilon}\right)d\nu\right]\eta\, d\mu \overset{!}{=} 0$$

set:
$$\exp\left(-\frac{\phi}{\varkappa} - \frac{\phi}{\varepsilon}\right) = \int \exp(\dots)\, d\nu$$

$$\phi = -\left(\frac{1}{\varkappa} + \frac{1}{\varepsilon}\right)^{-1} \log\left(\int \exp(\dots)\, d\nu\right)$$

$$= -\frac{\varkappa \varepsilon}{\varkappa + \varepsilon} \log\left(\int \exp(\dots)\, d\nu\right) \Rightarrow \text{only a minor modification}$$

evaluating the PD gap

$$\gamma = \exp\left(\frac{\phi \oplus \psi - c}{\varepsilon}\right) \mu \otimes \nu \qquad \sigma_i = P_i \gamma$$

primal: $\underbrace{\kappa\, KL(\sigma_1 | \mu)}_{(i)} + \underbrace{\kappa\, KL(\sigma_2 | \nu)}_{(ii)} + \underbrace{\int c\, d\gamma + \varepsilon KL(\gamma | \mu \otimes \nu)}_{(iii)}$

dual: $\underbrace{\kappa \int \left[1 - \exp(-\phi/\kappa)\right] d\mu}_{(id)} + \underbrace{\kappa \int \left[1 - \exp(-\psi/\kappa)\right] d\nu}_{(iid)} \underbrace{- \varepsilon \int \left[\exp\left(\frac{\phi \oplus \psi - c}{\varepsilon}\right) - 1\right] d\mu \otimes \nu}_{(iiid)}$

$$(iii) - (iii\text{-}d) = \int c\, d\gamma + \varepsilon \int \left(\frac{\phi \oplus \psi - c}{\varepsilon}\right) d\gamma \boxed{- \varepsilon \|\gamma\| + \varepsilon \|\mu \otimes \nu\| + \varepsilon \|\gamma\| - \varepsilon \|\mu \otimes \nu\|}$$

$$= \left[\int \phi\, d\sigma_1 + \int \psi\, d\sigma_2\right]$$

$$(i) - (id) = \kappa \left[\int \log\left(\frac{d\sigma_1}{d\mu}\right) d\sigma_1 - \|\sigma_1\| \boxed{+ \|\mu\| - \|\mu\|} + \int \exp(-\phi/\kappa)\, d\mu\right]$$